MATH 323 5 Probability Theory

3/2/23

Continuous r.v.

Examples of Expected Values and Variances

1) Uniform $[A, B]$

$$\mathbb{E}(X) = \frac{A+B}{2} \qquad Var(X) = \frac{(B-A)^2}{12}$$

2) Exponential $\quad f(x) = \lambda e^{-\lambda x}$

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$\mathbb{E}(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x} \, dx =$$

$y = \lambda x$

$$= \frac{1}{\lambda^2} \int_0^\infty y^2 e^{-\lambda y} \, dy =$$

$$= \frac{\Gamma(3)}{\lambda^2} = \frac{2}{\lambda^2}$$

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Normal (μ, σ^2)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mathbb{E}(X) = \mu$$

$$\mathbb{E}(X^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx =$$

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z - \mu)^2 e^{-\frac{z^2}{2}} dz =$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz \quad +$$

$$\frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}} dz \quad + \qquad = 0$$

$$\frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz \qquad = \mu^2$$

$$\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} \, dz = \int_{-\infty}^{\infty} z \; z e^{-\frac{z^2}{2}} \, dz =$$

$$-\frac{d}{dz} e^{-\frac{z^2}{2}} = z e^{-\frac{z^2}{2}}$$

$$= \underbrace{-z^2 e^{-\frac{z^2}{2}}}_{=0} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} \, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \, dz = 1$$

$$\mathbb{E}(X^2) = \sigma^2 + \mu^2$$

$$\mathrm{Var}(X) = \sigma^2$$

If X is $N(\mu, \sigma^2)$ Then

$\mathbb{E}(X) = \mu \qquad \mathrm{Var}(X) = \sigma^2$

A $\quad X \sim \mathcal{N}(1, 16)$

$Var(X) = 16 \qquad \sigma_X = 4$

U uniform in $[0, 1]$.

Generate X such that

X is uniform in $[A, B]$.

$$X = A + (B - A)U$$

$$\mathbb{P}(X \leq x) = \mathbb{P}(A + (B-A)U \leq x)$$

$$\mathbb{P}\left(U \leq \frac{x - A}{B - A}\right) =$$

$$x \geq B \implies \frac{x - A}{B - A} \geq 1$$

$$\mathbb{P}\left(U \leq \frac{x - A}{B - A}\right) = 1$$

$$x \leq A \implies \frac{x - A}{B - A} \leq 0$$

$$\mathbb{P}\left(U \leq \frac{x - A}{B - A}\right) = 0$$

$x \in [A, B]$

$$\mathbb{P}(X \leq x) = \mathbb{P}\left(U \leq \frac{x-A}{B-A}\right) =$$

$$\frac{x-A}{B-A}$$

$$F_X(x) = \begin{cases} 0 & x \leq A \\ \frac{x-A}{B-A} & A \leq x \leq B \\ 1 & x \geq B \end{cases}$$

I want X to be exponent.

U is uniform

$X = -\ln U$

$$\mathbb{P}(X \geq x) = \mathbb{P}(-\ln U \geq x) =$$

$$\mathbb{P}(\ln U \leq -x) =$$

$$\mathbb{P}(U \leq e^{-x}) = e^{-x}$$

$$F_X(x) = 1 - e^{-x}$$

If f is a p.d.f. with c.d.f. F and U is uniform in $[0,1]$ Then

$$F^{-1}(U) = X$$

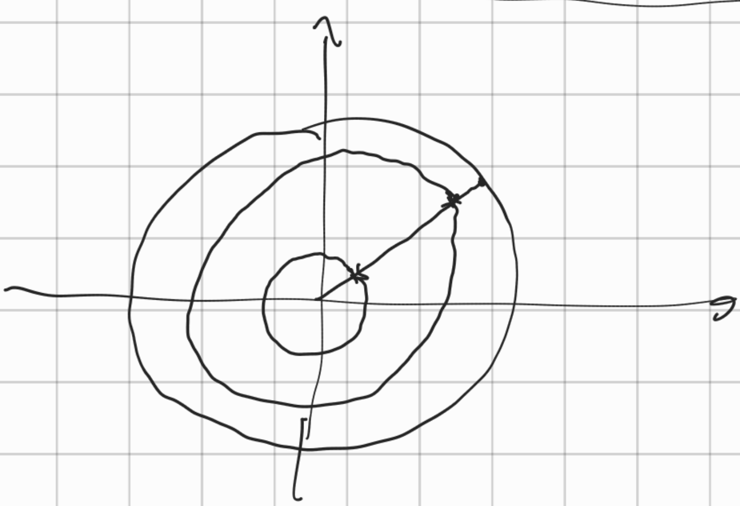
has p.d.f. $f(x)$.

$$\mathbb{P}(X \le x) = \mathbb{P}(F^{-1}(U) \le x) =$$

$$\mathbb{P}(U \le F(x)) = F(x)$$

$N(0,1)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$



(X, Y) uniform in The unit circle.

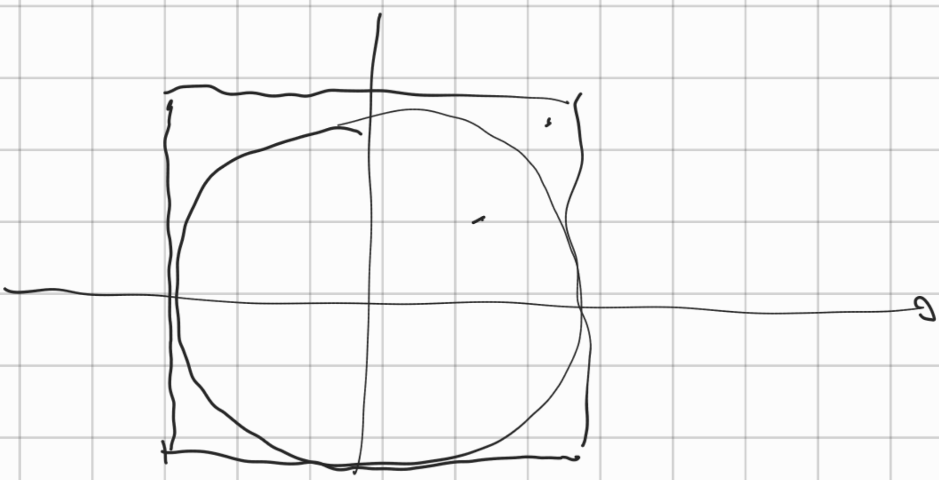
Choose $r \in [0,1]$ with density

$$f_R(r) = \frac{r}{2}$$

Choose $\theta \in [0, 2\pi]$ with density

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

$$(x, y) = (r\cos\theta, r\sin\theta)$$



pick a point (x,y) uniformly in the square.

If (x,y) is in the circle $\to$ done

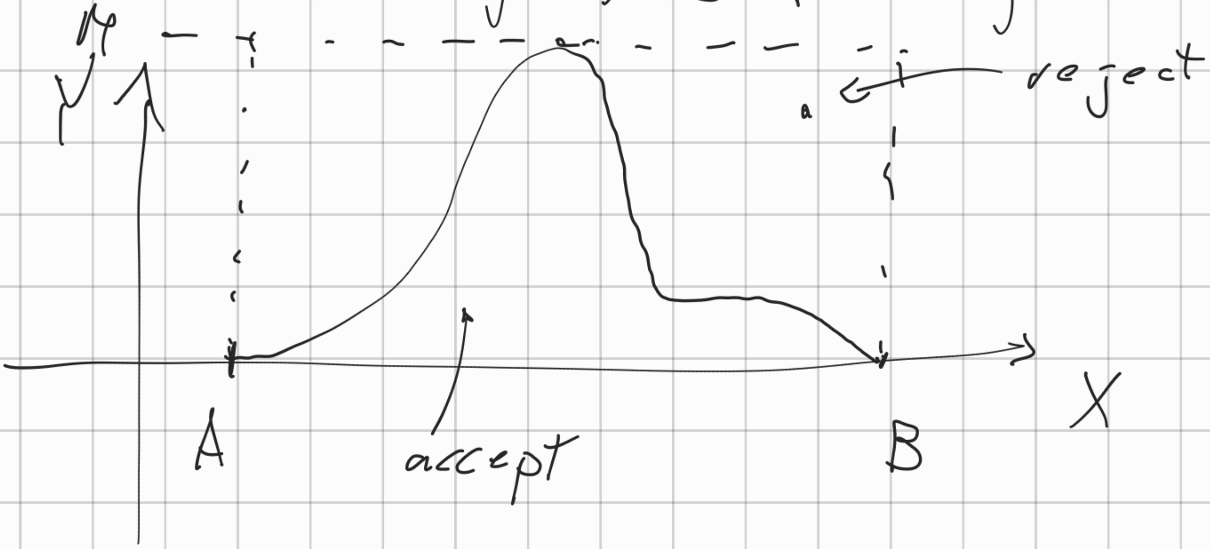
if (x,y) is not in the circle, choose another one.

Area square is 4

Area circle is π

I average you will select $\frac{4}{\pi}$ point before getting a good I.

X $\quad f(x) = 0$ outside an interval $[A, B]$ and $f(x) \leq M$ for some M



Generate X uniform in $[A, B]$

Y uniform in $[0, M]$.

if (X, Y) is below The graph

of f Then Y is your number

If not, repeat.

The condition (X,Y) is under

The graph $\Rightarrow$ $Y \leq f(X)$

X r.v. $h : \mathbb{R} \to \mathbb{R}$

$Y = h(X)$ is a r.v.

What is the p.d.f. of Y?

$$\mathbb{P}(Y \leq y) = \mathbb{P}(h(X) \leq y) =$$

$$\mathbb{P}(X \leq h^{-1}(y)) = F_X(h^{-1}(y))$$

$$\boxed{f_Y(y) = \frac{d}{dy} h^{-1}(y) \, f_X(h^{-1}(y))}$$

$$\mathbb{P}(a \leq X \leq b) =$$

$$= \int_a^b f_X(x)\, dx$$

$y = h(x)$

$$= \int_{h(a)}^{h(b)} \frac{d}{dy} h^{-1}(y)\, f_X(h^{-1}(y))\, dy =$$

$$\mathbb{P}(h(a) \le Y \le h(b))$$

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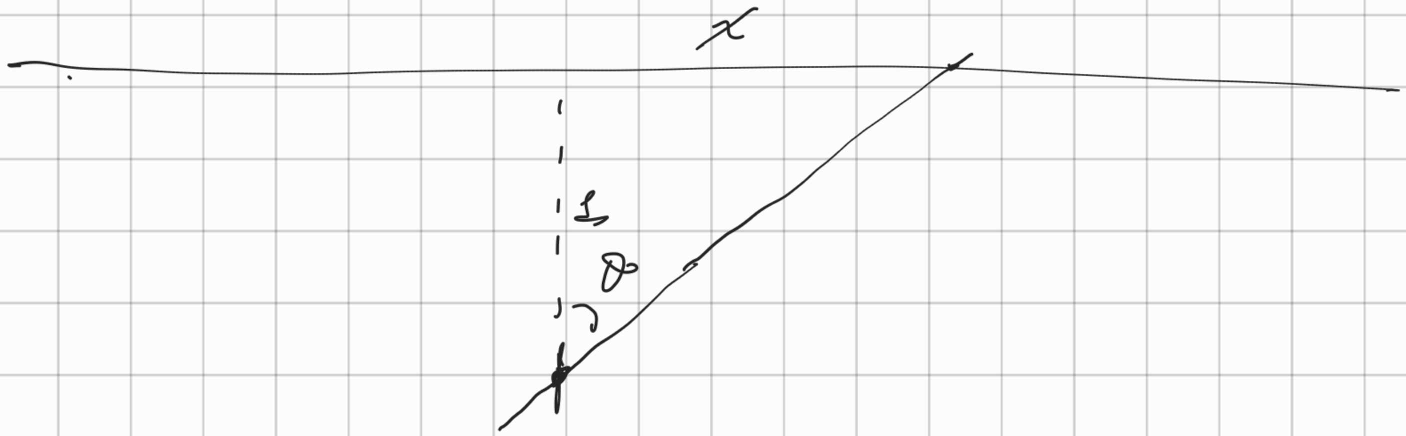
if X is $\mathcal{N}(\mu, \sigma^2)$

Then

$$Z = \frac{X - \mu}{\sigma} \quad \text{is} \quad \mathcal{N}(0,1)$$



p.d.f of X.

$$x = \frac{\ell}{2}\tan\theta$$

Θ

$$\mathbb{P}(X \le x) = \mathbb{P}(\tan\Theta \le x) =$$

$$= \mathbb{P}(\Theta \le \arctan x) =$$

$$= \frac{\arctan x + \frac{\pi}{2}}{\pi}$$

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$ Cauchy.